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Raising and lowering operators for $u_q(n)$

C Quesne†‡

Physique Nucléaire Théorique et Physique Mathématique, Université Libre de Bruxelles, Campus de la Plaine CP229, Bd. du Triomphe, B1050 Bruxelles, Belgium

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Abstract. After reviewing the defining relations for the q-algebra $u_q(n)$, we construct a Cartan–Weyl basis and list the q-commutation relations of its generators. We then use the latter to explicitly construct sets of raising and lowering operators for the canonical chain of q-algebras $u_q(n) \supset u_q(n-1)$, generalizing those introduced by Nagel and Moshinsky for $u(n) \supset u(n-1)$. Finally, we give their normalization coefficients and show how the normalized operators can be used to go from any Gel'fand–Tseitlin basis state that is of highest weight in $u_q(n-1)$ to any other one and ultimately to construct the whole Gel'fand–Tseitlin basis from its highest-weight state. This paper both generalizes a previous work of Ueno *et al* on lowering operators and provides explicit expressions for their operators in terms of $u_q(n)$ Cartan–Weyl generators.

1. Introduction

In recent years, there has been considerable interest in the so-called q-algebras or quantum groups $U_q(g)$, which are q-deformations of the universal enveloping algebra of the Lie algebras g (Jimbo 1985a, Drinfeld 1986). In the case where q is not a root of unity, it was proved by Lusztig (1988) and Rosso (1988) that for any finite-dimensional irreducible representation (irrep) of a simple Lie algebra g, there is an irrep of $U_q(g)$ that has the same dimension and the same weight spectrum, and so can be uniquely labelled by its highest weight. Hence the theory of q-algebra irreps bears much similarity to that of ordinary Lie algebras.

In this paper, we shall be concerned with the unitary irreps of $u_q(n) \equiv U_q(u(n))$ when $q \in \mathbb{R}^+$ (the results presented here could however be easily extended to the case where q is a phase different from a root of unity). Such irreps can be characterized by a Young diagram $[h_1 h_2 \ldots h_n]$, where h_i, $i = 1, \ldots, n$, are some integers satisfying the inequalities $h_1 \geqslant h_2 \geqslant \cdots \geqslant h_n$. One can define a basis of their carrier space, the so-called Gel'fand–Tseitlin (GT) basis (Gel'fand and Tseitlin 1950), whose vectors are completely specified by the irreps of the q-subalgebras of $u_q(n)$ belonging to the canonical chain $u_q(n) \supset u_q(n-1) \supset \cdots \supset u_q(1)$ (Jimbo 1985b, Ueno *et al* 1989).

Ueno *et al* (1989) showed that the GT basis can be constructed via a lowering operator method. Their operators are the q-analogues of the lowering operators that were introduced, together with raising operators, by Nagel and Moshinsky (1965a, b) to construct a GT basis for $u(n)$. The Nagel–Moshinsky $u(n) \supset u(n-1)$ lowering

† Directeur de recherches FNRS.
‡ E-mail address: cquesne@ulb.ac.be

(resp. raising) operators are some functions of the generators of a $u(n)$ Cartan–Weyl basis that, when acting on any GT basis vector, lower (resp. raise) some component of its weight by one unit. Furthermore, they transform any semi-maximal GT state, i.e. any GT basis vector that is of highest weight in $u(n-1)$, into another semi-maximal GT state. Hence, they enable one to obtain from the maximal or highest-weight state of a given irrep all the semi-maximal states belonging to its carrier space. By combining lowering operators of $u(n)$, $u(n-1), \ldots, u(2)$, one can then construct the full set of GT basis vectors from the maximal one.

The $u_q(n) \supset u_q(n-1)$ lowering operators considered by Ueno *et al* (1989) were only inductively defined and were not expressed in terms of $u_q(n)$ generators. When dealing with applications of $u_q(n)$ to physical models, it may be necessary however to construct the GT basis vectors in explicit form. For such purpose, one needs to know the lowering operators as functions of the generators. The aim of the present paper is to determine such functions or, in other words, to provide an explicit solution to the Ueno *et al* recursion relations. In addition, a similar problem will be solved for the $u_q(n) \supset u_q(n-1)$ raising operators, which were not considered by these authors.

In the following section, we review the defining relations for $u_q(n)$, construct a Cartan–Weyl basis and list the q-commutation relations of its generators. In section 3, we use the latter to explicitly construct sets of raising and lowering operators for $u_q(n) \supset u_q(n-1)$. In section 4, we normalize these operators and use them to pass from any semi-maximal GT basis vector to any other one. Finally, section 5 contains the conclusion.

2. Cartan–Chevalley and Cartan–Weyl bases of $u_q(n)$

The $u_q(n) \boxminus U_q(u(n))$ q-algebra, corresponding to a one-parameter deformation of the universal enveloping algebra of $u(n)$, is defined (Jimbo 1985a) as the associative algebra over $\mathbb{C}$ generated by I, E^i_i, $i = 1, 2, \ldots, n$, E^{i+1}_i, E^i_{i+1}, $i = 1, 2, \ldots, n-1$, and the commutation relations

$$[E^i_i, E^j_j] = 0 \tag{2.1a}$$

$$[E^i_i, E^{j+1}_j] = (\delta^i_j - \delta^{j+1}_i) E^{j+1}_j \tag{2.1b}$$

$$[E^i_i, E^j_{j+1}] = (\delta^i_{j+1} - \delta^j_i) E^j_{j+1} \tag{2.1c}$$

$$[E^{i+1}_i, E^j_{j+1}] = \delta^j_i [E^i_i - E^{i+1}_{i+1}] \tag{2.1d}$$

together with the quadratic and cubic q-Serre relations given by

$$[E^{i+1}_i, E^{j+1}_j] = 0 \qquad j \neq i \pm 1 \tag{2.2a}$$

$$[E^i_{i+1}, E^j_{j+1}] = 0 \qquad j \neq i \pm 1 \tag{2.2b}$$

and

$$(E^{i+1}_i)^2 E^{j+1}_j - [2] E^{i+1}_i E^{j+1}_j E^{i+1}_i + E^{j+1}_j (E^{i+1}_i)^2 = 0 \qquad j = i \pm 1 \tag{2.3a}$$

$$(E^i_{i+1})^2 E^j_{j+1} - [2] E^i_{i+1} E^j_{j+1} E^i_{i+1} + E^j_{j+1} (E^i_{i+1})^2 = 0 \qquad j = i \pm 1 \tag{2.3b}$$

respectively. The definition of the algebra is completed by assuming the Hermiticity properties

$$(E_i^i)^\dagger = E_i^i \qquad (E_i^{i+1})^\dagger = E_{i+1}^i . \tag{2.4}$$

Note that in (2.3), [2] denotes a q-number, whose general definition is

$$[n] \equiv \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} = q^{(n-1)/2} + q^{(n-3)/2} + \cdots + q^{-(n-1)/2} \qquad n \in \mathbb{Z} \qquad q \in \mathbb{R}^+ . \tag{2.5}$$

This definition of q-numbers is extended to the commuting operators $E_i^i - E_{i+1}^{i+1}$ in (2.1d).

The set of operators E_i^i, E_i^{i+1}, and E_{i+1}^i is the q-analogue of the Cartan–Chevalley basis of $u(n)$. The operators E_i^i, $i = 1,2,\ldots,n$, are the weight generators, obtained by combining the generators of the $su(n)$ Cartan subalgebra $E_i^i - E_{i+1}^{i+1}$, $i = 1,2,\ldots,n-1$, with an additional commuting generator $\sum_{i=1}^n E_i^i$. The operators E_i^{i+1}, $i = 1, 2, \ldots, n-1$, are the raising generators corresponding to the $su(n)$ simple roots α_i, whereas E_{i+1}^i, $i = 1,2,\ldots,n-1$, are the lowering generators associated with the $su(n)$ roots $-\alpha_i$.

To construct raising and lowering operators for $u_q(n)$, we shall need a q-analogue of the $u(n)$ Cartan–Weyl basis E_i^j, i, $j = 1,2,\ldots,n$. So we have to introduce additional raising and lowering generators E_i^{i+p}, E_{i+p}^i, for $p > 1$. There exist various conventions for these operators in the literature (see e.g. Rosso 1989, Chakrabarti 1991). Here we shall make the same assumptions as in Quesne (1991) and define them recursively by

$$E_i^{i+p} \equiv [E_i^{i+1}, E_{i+1}^{i+p}]_q \qquad i = 1,\ldots,n-2 \qquad p = 2,\ldots,n-i \tag{2.6a}$$

$$E_{i+p}^i \equiv [E_{i+p}^{i+1}, E_{i+1}^i]_{q^{-1}} \qquad i = 1,\ldots,n-2 \qquad p = 2,\ldots,n-i \tag{2.6b}$$

in terms of q-commutators of the type

$$[A, B]_{q^a} \equiv AB - q^{a/2} BA = -q^{a/2} [B, A]_{q^{-a}} . \tag{2.7}$$

The cubic q-Serre relations (2.3) can then be expressed in the form of q-commutators as

$$[E_i^{i+1}, E_i^{i+2}]_{q^{-1}} = [E_{i+1}^{i+2}, E_i^{i+2}]_q = 0 \tag{2.8a}$$

$$[E_{i+2}^i, E_{i+1}^i]_q = [E_{i+2}^i, E_{i+2}^{i+1}]_{q^{-1}} = 0 . \tag{2.8b}$$

In the $u(n)$ case, the Cartan–Weyl generators E_i^j satisfy the well known commutation relations

$$[E_i^j, E_k^l] = \delta_k^j E_i^l - \delta_i^l E_k^j \qquad (q = 1) . \tag{2.9}$$

The q-analogues of these relations can be written in terms of q-commutators (2.7). In their derivation, use is made of the most general form of the q-Jacobi identity (Chaichian *et al* 1990)

$$q^{b/2}[A, [B, C]_{q^a}]_{q^{c-b}} + q^{c/2}[B, [C, A]_{q^b}]_{q^{a-c}} + q^{a/2}[C, [A, B]_{q^c}]_{q^{b-a}} = 0 \tag{2.10}$$

where a, b, $c \in \mathbb{R}$, and of a symmetry property of the Cartan–Weyl basis given in the following lemma:

Table 1. q-commutators of two raising generators E_i^j, E_k^l or two lowering ones E_j^i, E_l^k in the case where $0 < j - i \leqslant l - k$.

Conditions	q^a	$[E_i^j, E_k^l]_{q^a}$	$[E_j^i, E_l^k]_{q^a}$
$i < j < k < l$	1	0	0
$i < j = k < l$	q	E_i^l	$-q^{1/2} E_l^i$
$i < k < j < l$	1	$-\left(q^{1/2} - q^{-1/2}\right) E_i^l E_k^j$	$-\left(q^{1/2} - q^{-1/2}\right) E_j^k E_l^i$
$i = k < j < l$	q^{-1}	0	0
$i = k < j = l$	1	0	0
$k < i < j < l$	1	0	0
$k < i < j = l$	q	0	0
$k < i < l < j$	1	$\left(q^{1/2} - q^{-1/2}\right) E_k^j E_i^l$	$\left(q^{1/2} - q^{-1/2}\right) E_l^i E_j^k$
$k < i = l < j$	q^{-1}	$-q^{-1/2} E_k^j$	E_j^k
$k < l < i < j$	1	0	0

Lemma 2.1. The generators of the $u_q(n)$ Cartan–Weyl basis are such that

$$\left(E_i^j\right)^\dagger_{q \to q^{-1}} = E_j^i \tag{2.11}$$

where, on the left-hand side, Hermitian conjugation is combined with the simultaneous change of q into q^{-1}.

Proof. Direct verification using (2.1)–(2.6). □

A first important result consists in a generalization of (2.6).

Lemma 2.2. The additional generators (2.6) of the Cartan–Weyl basis satisfy the relations

$$E_i^{i+p} \equiv \left[E_i^{i+r}, E_{i+r}^{i+p}\right]_q \qquad i = 1, \ldots, n-2 \qquad p = 2, \ldots, n-i \tag{2.12a}$$

$$E_{i+p}^i \equiv \left[E_{i+p}^{i+r}, E_{i+r}^i\right]_{q^{-1}} \qquad i = 1, \ldots, n-2 \qquad p = 2, \ldots, n-i \tag{2.12b}$$

for any $r \in \{1, 2, \ldots, p-1\}$.

Proof. Equation (2.12*a*) is proved by induction over r by starting from $r = 1$, for which it coincides with definition (2.6*a*). For such purpose, use is made of the q-Jacobi identity (2.10) for $a = c = 1$, $b = 0$, and of (2.2). Equation (2.12*b*) then follows from (2.11). □

We state the general results in the form of a theorem.

Theorem 2.3. The $u_q(n)$ Cartan–Weyl generators E_i^j, $i, j = 1, \ldots, n$, satisfy the q-commutation relations given in tables 1, 2, and

$$\left[E_i^i, E_j^k\right] = \left(\delta_j^i - \delta_i^k\right) E_j^k \tag{2.13}$$

as well as those that can be derived from table 1 by using (2.7).

Proof. For $k = j$, $j + 1$, and $j - 1$, (2.13) reduces to (2.1*a*)–(2.1*c*). For $k = j + p$, $p > 1$, it can be proved by induction over p by using (2.10) with $a = 1$, $b = c = 0$. For the remaining values of k, use is made of lemma 2.1.

Table 2. Non-vanishing commutators of a raising generator E_i^j, $i < j$, with a lowering one E_k^l, $k > l$.

Conditions	$[E_i^j, E_k^l]$
$i < l < j < k$	$\left(q^{1/2} - q^{-1/2}\right) q^{-(E_l^l - E_j^j)/2} E_i^l E_k^j$
$i < l < j = k$	$q^{-(E_l^l - E_j^j)/2} E_i^l$
$i = l < j < k$	$-q^{-(E_i^i - E_j^j)/2} E_k^j$
$i = l < j = k$	$\left[E_i^i - E_j^j\right]$
$i = l < k < j$	$-q^{(E_i^i - E_k^k + 1)/2} E_k^j$
$l < i < j = k$	$q^{(E_i^i - E_j^j - 1)/2} E_i^l$
$l < i < k < j$	$-\left(q^{1/2} - q^{-1/2}\right) q^{(E_i^i - E_k^k)/2} E_i^l E_k^j$

Consider next the q-commutators given in table 1. One first notices that by successively using the symmetries mentioned in lemmas 2.2 and 2.3, the q-commutators of two lowering generators listed in column 4 can be derived from those of two raising generators listed in column 3. It then only remains to consider the latter, which can be rewritten as $\left[E_i^{i+p}, E_k^{k+r}\right]_{q^a}$ by setting $j = i + p$, $l = k + r$, where $p \leqslant r$. The demonstration proceeds by a double induction over p and r, starting from $p = r = 1$, and is sketched in the appendix.

Finally, the commutators listed in table 2 can be proved in a similar way. By setting $j = i + p$ and $k = l + r$, they can be rewritten as $\left[E_i^{i+p}, E_{l+r}^l\right]$. By virtue of lemma 2.1, it is enough to consider the case $p \leqslant r$. One successively establishes the relations for $p = 1 \leqslant r$, $p = 2 \leqslant r$, and $2 < p \leqslant r$. Note that the case $p = 1 \leqslant r$ is special and cannot serve as the starting point of the induction over p because some rows of table 2 disappear. □

Remark. Contrary to what happens in the $u(n)$ case, some q-commutators (resp. commutators) of table 1 (resp. 2) are quadratic functions of the commuting Cartan–Weyl generators E_i^l, E_k^j or E_j^k, E_l^i. As a matter of fact, by using (2.10) one can show that no value of $a \in \mathbb{R}$ can be found so that they become linear functions as in the Lie algebra case.

3. Construction of raising and lowering operators

For a $u_q(n)$ unitary irrep characterized by its highest weight $[h_1 h_2 \ldots h_n]$, where $h_1 \geqslant h_2 \geqslant \cdots \geqslant h_n$, the GT basis of its carrier space is specified by a set of integers h_{ij}, $1 \leqslant i \leqslant j \leqslant n$, such that $h_{in} = h_i$, $i = 1, \ldots, n$, and

$$h_{i,j+1} \geqslant h_{ij} \geqslant h_{i+1,j+1} \qquad 1 \leqslant i \leqslant j \leqslant n - 1 . \tag{3.1}$$

They are arranged to form a triangular array, the GT pattern, so that the GT basis vectors are written as

$$|h_{ij}\rangle \equiv \left|\begin{matrix} h_{1n} & & h_{2n} & & \cdots & & h_{n-1,n} & & h_{nn} \\ & h_{1,n-1} & & h_{2,n-1} & \cdots & h_{n-2,n-1} & & h_{n-1,n-1} & \\ & & & & \cdots & & & & \\ & & & & h_{12} \quad h_{22} & & & & \\ & & & & h_{11} & & & & \end{matrix}\right\rangle . \tag{3.2}$$

Here $[h_{1m}h_{2m}\ldots h_{mm}]$ characterizes an irrep of the q-subalgebra $u_q(m)$ belonging to the canonical chain

$$u_q(n) \supset u_q(n-1) \supset \cdots \supset u_q(m) \supset \cdots \supset u_q(1) \tag{3.3}$$

and whose Cartan–Weyl generators are E_i^j, $i, j = 1, 2, \ldots, m$.

The weight of the state (3.2) is given by the set of eigenvalues

$$w_i = \sum_{k=1}^{i} h_{ki} - \sum_{k=1}^{i-1} h_{k,i-1} \tag{3.4}$$

of the generators E_i^i, $i = 1, \ldots, n$. The highest- and lowest-weight states (or maximal and minimal states) correspond to $h_{ij} = h_{in}$ and $h_{ij} = h_{i+1,n}$ for $1 \leqslant i \leqslant j \leqslant n-1$, respectively.

Let us denote the semi-maximal states, i.e. the GT basis vectors that are of highest weight in $u_q(n-1)$, by

$$\left|\begin{matrix} h_i \\ r_i \end{matrix}\right\rangle \equiv \left|\begin{matrix} h_1 & h_2 & & \cdots & & h_{n-1} & & h_n \\ & r_1 & r_2 & \cdots & r_{n-2} & & r_{n-1} & \\ & & & \cdots & & & & \\ & & & r_1 \quad r_2 & & & & \\ & & & r_1 & & & & \end{matrix}\right\rangle . \tag{3.5}$$

They satisfy the following relations

$$E_j^j \left|\begin{matrix} h_i \\ r_i \end{matrix}\right\rangle = r_j \left|\begin{matrix} h_i \\ r_i \end{matrix}\right\rangle \qquad 1 \leqslant j \leqslant n-1 \tag{3.6a}$$

$$E_n^n \left|\begin{matrix} h_i \\ r_i \end{matrix}\right\rangle = \left(\sum_{j=1}^{n} h_j - \sum_{j=1}^{n-1} r_j\right) \left|\begin{matrix} h_i \\ r_i \end{matrix}\right\rangle \tag{3.6b}$$

$$E_j^{j+1} \left|\begin{matrix} h_i \\ r_i \end{matrix}\right\rangle = 0 \qquad 1 \leqslant j < n-1 \tag{3.6c}$$

where, by virtue of (2.6a), the latter implies that

$$E_j^k \left|\begin{matrix} h_i \\ r_i \end{matrix}\right\rangle = 0 \qquad 1 \leqslant j < k \leqslant n-1 \,. \tag{3.7}$$

In particular, the highest-weight state corresponds to $r_i = h_i$, $i = 1, \ldots, n-1$, and will be denoted by $\left|\begin{smallmatrix} h_i \\ h_i \end{smallmatrix}\right\rangle$.

It is possible to go from any semi-maximal state to any other one contained in the same representation space by using lowering and raising operators. The latter are defined as follows:

Definition 3.1. A set of lowering operators L_n^m, $m = 1, 2, \ldots, n-1$, for $u_q(n) \supset u_q(n-1)$ is a set of functions of the $u_q(n)$ generators E_i^j, $i, j = 1, \ldots, n$, satisfying the two conditions

$$[E_i^i, L_n^m] = -\delta_i^m L_n^m \qquad 1 \leqslant i, m < n \tag{3.8}$$

$$L_n^m \left| \begin{matrix} h_i \\ r_i \end{matrix} \right\rangle = N_{r_i - \delta_{im}}^{r_i} \left| \begin{matrix} h_i \\ r_i - \delta_{im} \end{matrix} \right\rangle \qquad 1 \leqslant m < n \tag{3.9}$$

where

$$N_{r_i - \delta_{im}}^{r_i} \equiv N_{r_1 \ldots r_{m-1} r_m - 1 r_{m+1} \ldots r_{n-1}}^{r_1 \ldots r_{m-1} r_m r_{m+1} \ldots r_{n-1}} \tag{3.10}$$

is some normalization coefficient. A set of raising operators R_m^n, $m = 1, 2, \ldots, n-1$, is similarly defined by replacing (3.8)–(3.10) by

$$[E_i^i, R_m^n] = \delta_m^i R_m^n \qquad 1 \leqslant i, m < n \tag{3.11}$$

$$R_m^n \left| \begin{matrix} h_i \\ r_i \end{matrix} \right\rangle = N_{r_i + \delta_{im}}^{r_i} \left| \begin{matrix} h_i \\ r_i + \delta_{im} \end{matrix} \right\rangle \qquad 1 \leqslant m < n \tag{3.12}$$

$$N_{r_i + \delta_{im}}^{r_i} \equiv N_{r_1 \ldots r_{m-1} r_m + 1 r_{m+1} \ldots r_{n-1}}^{r_1 \ldots r_{m-1} r_m r_{m+1} \ldots r_{n-1}} \tag{3.13}$$

respectively.

Remarks. (1) Equation (3.8) (resp. (3.11)) means that L_n^m (resp. R_m^n) when acting on any GT state $|h_{ij}\rangle$ of weight w_i, $1 \leqslant i < n$, given in (3.4), with respect to $u_q(n-1)$, lowers (resp. raises) the mth component of the weight by 1, i.e. gives an arbitrary linear combination of GT states $|h'_{ij}\rangle$ of weight $w'_i = w_i - \delta_{im}$ (resp. $w'_i = w_i + \delta_{im}$), $1 \leqslant i < n$. Equation (3.9) (resp. (3.12)) imposes that whenever $|h_{ij}\rangle = \left| \begin{smallmatrix} h_i \\ r_i \end{smallmatrix} \right\rangle$, the lowered (resp. raised) weight w'_i becomes the highest weight of a $u_q(n-1)$ irrep $[r_1 \ldots r_m - 1 \ldots r_{n-1}]$ (resp. $[r_1 \ldots r_m + 1 \ldots r_{n-1}]$). In other words, (3.9) and (3.12) are equivalent to the conditions

$$E_j^{j+1} L_n^m \left| \begin{matrix} h_i \\ r_i \end{matrix} \right\rangle = 0 \qquad 1 \leqslant j < n-1 \tag{3.14}$$

and

$$E_j^{j+1} R_m^n \left| \begin{matrix} h_i \\ r_i \end{matrix} \right\rangle = 0 \qquad 1 \leqslant j < n-1 \tag{3.15}$$

respectively.

(2) The sets of lowering and raising operators are not unique because from any lowering (resp. raising) operator one can construct another lowering (resp. raising) operator by adding terms containing raising generators on the right.

(3) We do not demand that L_n^m and R_m^n be *polynomial* functions of the generators as is done for $u(n)$. As we shall see below, in general they do not have such a property. We impose however as an extra condition that, in the limit $q \to 1$, they go over into the polynomial functions constructed for $u(n)$ by Nagel and Moshinky (1965a, b).

Let us now construct sets of raising and lowering operators.

Definition 3.2. Let the operators L^m_n, $\overline{L}^m_n$, R^n_m, $\overline{R}^n_m$, $1 \leqslant m < n$, and $R^{n'}_m$, $\overline{R}^{n'}_m$, $1 \leqslant n' < m < n$, be defined by

$$L^m_n = \left[\sum_{p=0}^{n-m-1} \sum_{i_p > i_{p-1} > \cdots > i_1 = m+1}^{n-1} E^m_{i_1} E^{i_1}_{i_2} \ldots E^{i_{p-1}}_{i_p} E^{i_p}_n \left(\prod_{\alpha=1}^{p} q^{-\mathcal{E}_{mi_\alpha}/2} [\mathcal{E}_{mi_\alpha}]^{-1} \right) \right] \times \prod_{j=m+1}^{n-1} [\mathcal{E}_{mj}] \qquad 1 \leqslant m < n \tag{3.16a}$$

$$\overline{L}^m_n = \left(\prod_{j=m+1}^{n-1} [\mathcal{E}_{mj}] \right) \sum_{p=0}^{n-m-1} q^{(n-m-p-1)/2} \sum_{i_p > i_{p-1} > \cdots > i_1 = m+1}^{n-1} \left(\prod_{\alpha=1}^{p} q^{-\mathcal{E}_{mi_\alpha}/2} [\mathcal{E}_{mi_\alpha}]^{-1} \right) \times E^{i_p}_n E^{i_{p-1}}_{i_p} \ldots E^{i_1}_{i_2} E^m_{i_1} \qquad 1 \leqslant m < n \tag{3.16b}$$

$$R^n_m = \left[E^n_m + \sum_{p=1}^{m-1} \sum_{i_p > i_{p-1} > \cdots > i_1 = 1}^{m-1} E^{i_p}_m E^{i_{p-1}}_{i_p} \ldots E^{i_1}_{i_2} E^n_{i_1} \left(q^{(m-p-i_1)/2} [\mathcal{E}_{mi_1}]^{-1} \right) \times \left(\prod_{\alpha=2}^{p} q^{-\mathcal{E}_{mi_\alpha}/2} [\mathcal{E}_{mi_\alpha}]^{-1} \right) \right] \prod_{j=1}^{m-1} [\mathcal{E}_{mj}] \qquad 1 \leqslant m < n \tag{3.17a}$$

$$\overline{R}^n_m = \left(\prod_{j=1}^{m-1} [\mathcal{E}_{mj}] \right) \left[q^{-(m-1)/2} E^n_m + \sum_{p=1}^{m-1} \sum_{i_p > i_{p-1} > \cdots > i_1 = 1}^{m-1} \left(q^{(1-i_1)/2} [\mathcal{E}_{mi_1}]^{-1} \right) \times \left(\prod_{\alpha=2}^{p} q^{-\mathcal{E}_{mi_\alpha}/2} [\mathcal{E}_{mi_\alpha}]^{-1} \right) E^n_{i_1} E^{i_1}_{i_2} \ldots E^{i_{p-1}}_{i_p} E^{i_p}_m \right] \qquad 1 \leqslant m < n \tag{3.17b}$$

$$R^{n'}_m = \left[\sum_{p=0}^{m-n'-1} q^{(m-n'-p-1)/2} \sum_{i_p > i_{p-1} > \cdots > i_1 = n'+1}^{m-1} E^{i_p}_m E^{i_{p-1}}_{i_p} \ldots E^{i_1}_{i_2} E^{n'}_{i_1} \times \left(\prod_{\alpha=1}^{p} q^{-\mathcal{E}_{mi_\alpha}/2} [\mathcal{E}_{mi_\alpha}]^{-1} \right) \right] \prod_{j=n'+1}^{m-1} [\mathcal{E}_{mj}] \qquad 1 \leqslant n' < m < n \tag{3.18a}$$

$$\overline{R}^{n'}_m = \left(\prod_{j=n'+1}^{m-1} [\mathcal{E}_{mj}] \right) \sum_{p=0}^{m-n'-1} \sum_{i_p > i_{p-1} > \cdots > i_1 = n'+1}^{m-1} \left(\prod_{\alpha=1}^{p} q^{-\mathcal{E}_{mi_\alpha}/2} [\mathcal{E}_{mi_\alpha}]^{-1} \right) \times E^{n'}_{i_1} E^{i_1}_{i_2} \ldots E^{i_{p-1}}_{i_p} E^{i_p}_m \qquad 1 \leqslant n' < m < n \tag{3.18b}$$

where

$$\mathcal{E}_{jk} = -\mathcal{E}_{kj} \equiv E^j_j - E^k_k + k - j \qquad 1 \leqslant j,k < n \tag{3.19}$$

and we assume that

$$\prod_{j=n}^{n-1}[\mathcal{E}_{mj}] = \prod_{j=1}^{0}[\mathcal{E}_{mj}] = \prod_{j=m}^{m-1}[\mathcal{E}_{mj}] = 1\,. \tag{3.20}$$

Remark. Note that from (3.6*a*), we obtain

$$\mathcal{E}_{jk}\begin{vmatrix} h_i \\ r_i \end{matrix}\Big\rangle = r_{jk}\begin{vmatrix} h_i \\ r_i \end{matrix}\Big\rangle \qquad r_{jk} = -r_{kj} \equiv r_j - r_k + k - j \qquad 1 \leqslant j,k < n\,. \tag{3.21}$$

The operators just introduced satisfy various properties that we shall now proceed to list. We first state some preliminary results in the form of lemmas.

Lemma 3.3. For any n' or n'' in the range indicated, the operators defined in (3.16*a*)–(3.18*b*) satisfy the recursion relations

$$L_n^m = \sum_{i=n'}^{n-1} L_i^m E_n^i q^{-\mathcal{E}_{mi}/2} \prod_{j=i+1}^{n-1} [\mathcal{E}_{mj}] + \left[E_n^{n'}, L_{n'}^m\right]_{q^{\sigma_{mn'}}} \prod_{j=n'}^{n-1} [\mathcal{E}_{mj}]$$
$$1 \leqslant m \leqslant n' < n \tag{3.22a}$$

$$\overline{L}_n^m = \sum_{i=n'}^{n-1} q^{(n-i-1)/2}\left(\prod_{j=i+1}^{n-1} [\mathcal{E}_{mj}]\right) q^{-\mathcal{E}_{mi}/2} E_n^i \overline{L}_i^m$$
$$+ q^{(n-n')/2}\left(\prod_{j=n'}^{n-1} [\mathcal{E}_{mj}]\right)\left[E_n^{n'}, \overline{L}_{n'}^m\right]_{q^{\sigma_{mn'}}} \qquad 1 \leqslant m \leqslant n' < n \tag{3.22b}$$

$$R_m^n = \sum_{i=1}^{n'} R_m^i E_i^n \prod_{j=1}^{i-1}[\mathcal{E}_{mj}] + \left[R_m^{n'}, E_{n'}^n\right] q^{(\mathcal{E}_{mn'}+1)/2} \prod_{j=1}^{n'}[\mathcal{E}_{mj}]$$
$$1 \leqslant n' \leqslant m < n \tag{3.23a}$$

$$\overline{R}_m^n = \sum_{i=1}^{n'} q^{-(i-1)/2}\left(\prod_{j=1}^{i-1}[\mathcal{E}_{mj}]\right) E_i^n \overline{R}_m^i + \left(\prod_{j=1}^{n'}[\mathcal{E}_{mj}]\right) q^{(\mathcal{E}_{mn'}-n')/2}\left[\overline{R}_m^{n'}, E_{n'}^n\right]$$
$$1 \leqslant n' \leqslant m < n \tag{3.23b}$$

$$R_m^{n'} = \sum_{i=n'+1}^{n''} q^{(i-n'-1)/2} R_m^i E_i^{n'} q^{-\mathcal{E}_{mi}/2} \prod_{j=n'+1}^{i-1} [\mathcal{E}_{mj}] + q^{(n''-n')/2}$$
$$\times \left[R_m^{n''}, E_{n''}^{n'}\right]_{q^{\sigma_{n''m}}} \prod_{j=n'+1}^{n''} [\mathcal{E}_{mj}] \qquad 1 \leqslant n' < n'' \leqslant m < n \tag{3.24a}$$

$$\overline{R}_m^{n'} = \sum_{i=n'+1}^{n''} \left(\prod_{j=n'+1}^{i-1} [\mathcal{E}_{mj}]\right) q^{-\mathcal{E}_{mi}/2} E_i^{n'} \overline{R}_m^i + \left(\prod_{j=n'+1}^{n''} [\mathcal{E}_{mj}]\right)\left[\overline{R}_m^{n''}, E_{n''}^{n'}\right]_{q^{\sigma_{n''m}}}$$
$$1 \leqslant n' < n'' \leqslant m < n \tag{3.24b}$$

respectively, where we define

$$L_m^m \equiv 1 \tag{3.25a}$$
$$\overline{L}_m^m \equiv 1 \tag{3.25b}$$
$$R_m^m \equiv 1 \tag{3.25c}$$
$$\overline{R}_m^m \equiv 1 \tag{3.25d}$$

and

$$\sigma_{ab} = \begin{cases} -1 & \text{if } a < b \\ 0 & \text{if } a = b\,. \end{cases} \tag{3.26}$$

Proof. The demonstration is similar to that for the corresponding $u(n)$ operators, except for the use of the q-commutation relations established in section 2 instead of (2.9). □

Remark. Note that in (3.23*a*) (resp. (3.23*b*)), the operators (3.17*a*) (resp. (3.17*b*)) are expressed in terms of the operators (3.18*a*) (resp. (3.18*b*)), in contrast to the remaining equations where the operators appearing on both sides belong to the same type.

In addition, we have the following property:

Lemma 3.4. The operators defined in (3.16)–(3.18) are such that

$$L_n^m = \overline{L}_n^m \qquad R_m^n = \overline{R}_m^n \qquad 1 \leqslant m < n \tag{3.27a}$$
$$R_m^{n'} = \overline{R}_m^{n'} \qquad 1 \leqslant n' < m < n\,. \tag{3.27b}$$

Proof. Choosing $n' = n-1$, $n' = 1$, and $n'' = n'+1$ in (3.22*a*), (3.23*a*), and (3.24*a*) respectively, we obtain the recursion relations

$$L_n^m = E_n^{n-1} L_{n-1}^m [\mathcal{E}_{m,n-1}] - [\mathcal{E}_{m,n-1}] L_{n-1}^m E_n^{n-1} \qquad 1 \leqslant m < n-1 \tag{3.28}$$
$$R_m^n = [\mathcal{E}_{m1}] R_m^1 E_1^n q^{\mathcal{E}_{m1}/2} - q^{\mathcal{E}_{m1}/2} E_1^n R_m^1 [\mathcal{E}_{m1}] \qquad 1 < m < n \tag{3.29}$$
$$R_m^{n'} = [\mathcal{E}_{m,n'+1}] R_m^{n'+1} E_{n'+1}^{n'} - E_{n'+1}^{n'} R_m^{n'+1} [\mathcal{E}_{m,n'+1}] \qquad 1 < n'+1 < m < n\,. \tag{3.30}$$

The same choices made in (3.22*b*), (3.23*b*), and (3.24*b*) lead to recursion relations coinciding with (3.28)–(3.30). Since, from (3.16) and (3.18), we get

$$L_{m+1}^m = E_{m+1}^m \tag{3.31a}$$
$$\overline{L}_{m+1}^m = E_{m+1}^m \tag{3.31b}$$
$$R_m^{m-1} = E_m^{m-1} \tag{3.31c}$$
$$\overline{R}_m^{m-1} = E_m^{m-1} \tag{3.31d}$$

respectively, by induction it follows from (3.28), (3.30), and their analogues for $\overline{L}_n^m$ and $\overline{R}_m^{n'}$ that equation (3.27*b*) and the first part of (3.27*a*) are valid. The second part of the latter then results from (3.27*b*) and the recursion relation (3.29) as well as its analogue for $\overline{R}_m^n$. □

We now state the main result of this section in the form of a theorem.

Theorem 3.5. The operators defined in (3.16) and (3.17) form sets of lowering and raising operators for $u_q(n)$ respectively.

Proof. By virtue of Lemma 3.4, it is enough to show that the operators (3.16*a*) (resp. (3.17*a*)) satisfy (3.8) and (3.14) (resp. (3.11) and (3.15)). The first equation is obviously fulfilled, while the second can be proved in the same way as the corresponding result for $u(n)$ (Nagel and Moshinsky 1965a) by using the q-commutation relations of section 2. In addition, it is easy to check that, when $q \to 1$, the operators (3.16) and (3.17) become the standard Nagel–Moshinsky operators. □

Remarks. (1) Although they appear in the recursion relations (3.23) of the raising operators $R^n_m = \overline{R}^n_m$, $1 \leqslant m < n$, the operators $R^{n'}_m = \overline{R}^{n'}_m$, $1 \leqslant n' < m < n$, defined in (3.18), are not raising operators of $u_q(n')$ as the notation might suggest. In fact, they are expressed exclusively in terms of weight and lowering generators.

(2) The lowering operators considered here coincide with those introduced by Ueno *et al* (1989). In fact, the latter use (3.25*a*), (3.28), and (3.31a) to inductively define lowering operators, then prove that (3.8) and (3.9) (i.e. our defining relations for lowering operators) are satisfied by such operators. Equation (3.16) therefore gives an explicit solution to the Ueno *et al* recursion relations.

We conclude the present section by listing the lowering and raising operators found for some small n values:

$u_q(2)$:

$$L^1_2 = E^1_2 \qquad R^2_1 = E^2_1 \tag{3.32}$$

$u_q(3)$:

$$L^1_3 = E^1_3[\mathcal{E}_{12}] + E^1_2E^2_3q^{-\mathcal{E}_{12}/2} = q^{1/2}[\mathcal{E}_{12}]E^1_3 + q^{-\mathcal{E}_{12}/2}E^2_3E^1_2 \tag{3.33a}$$

$$L^2_3 = E^2_3 \tag{3.33b}$$

$$R^3_1 = E^3_1 \tag{3.33c}$$

$$R^3_2 = E^3_2[\mathcal{E}_{21}] + E^1_2E^3_1 = q^{-1/2}[\mathcal{E}_{21}]E^3_2 + E^3_1E^1_2 \tag{3.33d}$$

$u_q(4)$:

$$\begin{aligned} L^1_4 = {} & E^1_4[\mathcal{E}_{12}][\mathcal{E}_{13}] + E^1_2E^2_4q^{-\mathcal{E}_{12}/2}[\mathcal{E}_{13}] + E^1_3E^3_4q^{-\mathcal{E}_{13}/2}[\mathcal{E}_{12}] + E^1_2E^2_3E^3_4q^{-(\mathcal{E}_{12}+\mathcal{E}_{13}/2)} \\ = {} & q[\mathcal{E}_{12}][\mathcal{E}_{13}]E^1_4 + q^{1/2}q^{-\mathcal{E}_{12}/2}[\mathcal{E}_{13}]E^2_4E^1_2 \\ & + q^{1/2}[\mathcal{E}_{12}]q^{-\mathcal{E}_{13}/2}E^3_4E^1_3 + q^{-(\mathcal{E}_{12}+\mathcal{E}_{13})/2}E^3_4E^2_3E^1_2 \end{aligned} \tag{3.34a}$$

$$L^2_4 = E^2_4[\mathcal{E}_{23}] + E^2_3E^3_4q^{-\mathcal{E}_{23}/2} = q^{1/2}[\mathcal{E}_{23}]E^2_4 + q^{-\mathcal{E}_{23}/2}E^3_4E^2_3 \tag{3.34b}$$

$$L^3_4 = E^3_4 \tag{3.34c}$$

$$R^4_1 = E^4_1 \tag{3.34d}$$

$$R^4_2 = E^4_2[\mathcal{E}_{21}] + E^1_2E^4_1 = q^{-1/2}[\mathcal{E}_{21}]E^4_2 + E^4_1E^1_2 \tag{3.34e}$$

$$\begin{aligned} R^4_3 = {} & E^4_3[\mathcal{E}_{31}][\mathcal{E}_{32}] + q^{1/2}E^1_3E^4_1[\mathcal{E}_{32}] + E^2_3E^4_2[\mathcal{E}_{31}] + E^2_3E^1_2E^4_1q^{-\mathcal{E}_{32}/2} \\ = {} & q^{-1}[\mathcal{E}_{31}][\mathcal{E}_{32}]E^4_3 + [\mathcal{E}_{32}]E^4_1E^1_3 + q^{-1/2}[\mathcal{E}_{31}]E^4_2E^2_3 + q^{-\mathcal{E}_{32}/2}E^4_1E^1_2E^2_3 . \end{aligned} \tag{3.34f}$$

4. Normalization of the raising and lowering operators

To completely characterize the lowering and raising operators just constructed, it remains to give the normalization coefficients appearing in equations (3.9) and (3.12). The former were already calculated by Ueno *et al* (1989), while the latter can be determined from them by using the following symmetry relation:

Lemma 4.1. The normalization coefficients of the raising operators are related with those of the lowering ones by the equation

$$N^{r_i}_{r_i+\delta_{im}} = \left(\prod_{j=1}^{m-1}[r_{mj}] \Big/ \prod_{j=m+1}^{n-1}[r_{mj}+1]\right) N^{r_i+\delta_{im}}_{r_i}. \tag{4.1}$$

Proof. From (3.9), (3.16*a*) and the Hermitian conjugate of (3.6*c*), it results that

$$N^{r_i}_{r_i-\delta_{im}} = \left\langle \begin{matrix} h_i \\ r_i-\delta_{im} \end{matrix} \middle| L^m_n \middle| \begin{matrix} h_i \\ r_i \end{matrix} \right\rangle = \left\langle \begin{matrix} h_i \\ r_i-\delta_{im} \end{matrix} \middle| E^m_n \middle| \begin{matrix} h_i \\ r_i \end{matrix} \right\rangle \prod_{j=m+1}^{n-1}[r_{mj}]. \tag{4.2}$$

By proceeding in the same way for the raising generators, we get

$$N^{r_i}_{r_i+\delta_{im}} = \left\langle \begin{matrix} h_i \\ r_i+\delta_{im} \end{matrix} \middle| R^n_m \middle| \begin{matrix} h_i \\ r_i \end{matrix} \right\rangle = \left\langle \begin{matrix} h_i \\ r_i+\delta_{im} \end{matrix} \middle| E^n_m \middle| \begin{matrix} h_i \\ r_i \end{matrix} \right\rangle \prod_{j=1}^{m-1}[r_{mj}]. \tag{4.3}$$

Equation (2.11), combined with the fact that $N^{r_i}_{r_i-\delta_{im}}$ can be expressed in terms of q-numbers (Ueno *et al* 1989) and is therefore invariant under $q \to q^{-1}$, then directly leads to (4.1). □

Hence we conclude that:

Theorem 4.2. The normalization coefficients $N^{r_i}_{r_i-\delta_{im}}$ and $N^{r_i}_{r_i+\delta_{im}}$ of (3.9) and (3.12) are given by

$$N^{r_i}_{r_i-\delta_{im}} = \left[-\left(\prod_{i=m+1}^{n-1}[r_{mi}] \Big/ \prod_{i=1}^{m-1}[r_{mi}-1]\right)\prod_{i=1}^{n}[r_m-h_i+i-m-1]\right]^{1/2} \tag{4.4a}$$

$$N^{r_i}_{r_i+\delta_{im}} = (-1)^{m-1}\left[-\left(\prod_{i=1}^{m-1}[r_{mi}] \Big/ \prod_{i=m+1}^{n-1}[r_{mi}+1]\right)\prod_{i=1}^{n}[r_m-h_i+i-m]\right]^{1/2} \tag{4.4b}$$

where r_{mi} is defined in (3.21).

The normalized raising and lowering operators can now be used to go from one semi-maximal state to another. In general this can be achieved along various paths. The following lemma proves their equivalence.

Lemma 4.3. The lowering and raising operators, defined in (3.16) and (3.17) respectively, satisfy the relations

$$[L_n^m, L_n^{m'}]\left|\begin{matrix}h_i\\r_i\end{matrix}\right\rangle = [R_m^n, R_{m'}^n]\left|\begin{matrix}h_i\\r_i\end{matrix}\right\rangle = [R_m^n, L_n^{m'}]\left|\begin{matrix}h_i\\r_i\end{matrix}\right\rangle = [R_{m'}^n, L_n^m]\left|\begin{matrix}h_i\\r_i\end{matrix}\right\rangle = 0$$
$$1 \leqslant m < m' < n\,. \tag{4.5}$$

Proof. By starting from (4.2) and using (3.16*a*) for $L_n^{m'}$ as well as the q-commutation relations of section 2, one easily obtains

$$\left\langle\begin{matrix}h_i\\r_i\end{matrix}\right|L_n^m L_n^{m'}\left|\begin{matrix}h_i\\r_i+\delta_{im}+\delta_{im'}\end{matrix}\right\rangle = \left\langle\begin{matrix}h_i\\r_i\end{matrix}\right|E_n^m E_n^{m'}\left|\begin{matrix}h_i\\r_i+\delta_{im}+\delta_{im'}\end{matrix}\right\rangle$$
$$\times\left(\prod_{j=m+1}^{n-1}[r_{mj}+1]\right)\left(\prod_{j=m'+1}^{n-1}[r_{m'j}+1]\right) \qquad 1 \leqslant m < m' < n\,. \tag{4.6}$$

A similar calculation for

$$\left\langle\begin{matrix}h_i\\r_i\end{matrix}\right|L_n^{m'} L_n^{m}\left|\begin{matrix}h_i\\r_i+\delta_{im}+\delta_{im'}\end{matrix}\right\rangle$$

leads to the same result, thus proving the first part of (4.5). The remaining parts can be demonstrated in a similar way. □

Remarks. (1) As in the $u(n)$ case, one has

$$[R_m^n, L_n^m]\left|\begin{matrix}h_i\\r_i\end{matrix}\right\rangle \neq 0 \qquad 1 \leqslant m < n. \tag{4.7}$$

(2) The first part of the lemma, relative to the lowering operators, was already stated by Ueno *et al* (1989).

By using lemma 4.3, we now obtain:

Lemma 4.4. Any semi-maximal state $\left|\begin{smallmatrix}h_i\\r_i'\end{smallmatrix}\right\rangle$ can be expressed in terms of any other one $\left|\begin{smallmatrix}h_i\\r_i\end{smallmatrix}\right\rangle$ as

$$\left|\begin{matrix}h_i\\r_i'\end{matrix}\right\rangle = \left(N_{r_1'r_2'\ldots r_{n-1}'}^{r_1r_2\ldots r_{n-1}}\right)^{-1}\left(\prod_{m=1}^{n-1} O_{r_m'}^{r_m}\right)\left|\begin{matrix}h_i\\r_i\end{matrix}\right\rangle \tag{4.8}$$

where the operators $O_{r_m'}^{r_m}$, $m = 1, \ldots, n-1$, are defined by

$$O_{r_m'}^{r_m} \equiv \begin{cases}(L_n^m)^{r_m - r_m'} & \text{if } r_m' < r_m\\ 1 & \text{if } r_m' = r_m\\ (R_m^n)^{r_m' - r_m} & \text{if } r_m' > r_m\end{cases} \tag{4.9}$$

and $N_{r_1'r_2'\ldots r_{n-1}'}^{r_1r_2\ldots r_{n-1}}$ is a normalization coefficient independent of the order of the operators $O_{r_m'}^{r_m}$.

Remark. The normalization coefficient in (4.8) satisfies the relation

$$N^{r_1 r_2 \ldots r_{n-1}}_{r'_1 r'_2 \ldots r'_{n-1}} = N^{r''_1 r''_2 \ldots r''_{n-1}}_{r'_1 r'_2 \ldots r'_{n-1}} N^{r_1 r_2 \ldots r_{n-1}}_{r''_1 r''_2 \ldots r''_{n-1}} \tag{4.10}$$

for any r''_1, $r''_2, \ldots, r''_{n-1}$ such that either $r'_m \leqslant r''_m \leqslant r_m$ or $r'_m \geqslant r''_m \geqslant r_m$, $1 \leqslant m < n$.

The following lemma gives the value of this normalization coefficient in two important special cases.

Lemma 4.5. The general normalization coefficients of the lowering and raising operators are given by

$$N^{r_1 r_2 \ldots r_{n-1}}_{r'_1 r'_2 \ldots r'_{n-1}} = \left[\left(\prod_{j>i=1}^{n-1} \frac{[r_i - r_j + j - i]!}{[r'_i - r'_j + j - i]!}\right)\left(\prod_{j\geqslant i=1}^{n-1} \frac{[h_i - r'_j + j - i]!}{[h_i - r_j + j - i]!}\right)\right.$$
$$\left. \times \left(\prod_{j>i=1}^{n} \frac{[r_i - h_j + j - i - 1]!}{[r'_i - h_j + j - i - 1]!}\right)\right]^{1/2} \tag{4.11}$$

where $r'_i \leqslant r_i$, $i = 1, \ldots, n-1$, and

$$N^{r_1 r_2 \ldots r_{n-1}}_{r'_1 r'_2 \ldots r'_{n-1}} = (-1)^{\sum_{i=1}^{n-1}(i-1)(r'_i - r_i)} \left[\left(\prod_{j>i=1}^{n-1} \frac{[r_i - r_j + j - i]!}{[r'_i - r'_j + j - i]!}\right)\right.$$
$$\left. \times \left(\prod_{j\geqslant i=1}^{n-1} \frac{[h_i - r_j + j - i]!}{[h_i - r'_j + j - i]!}\right)\left(\prod_{j>i=1}^{n} \frac{[r'_i - h_j + j - i - 1]!}{[r_i - h_j + j - i - 1]!}\right)\right]^{1/2} \tag{4.12}$$

where $r'_i \geqslant r_i$, $i = 1, \ldots, n-1$, respectively.

Remark. In particular, for $r_i = h_i$, equation (4.11) becomes

$$N^{h_1 h_2 \ldots h_{n-1}}_{r_1 r_2 \ldots r_{n-1}} = \left[\left(\prod_{j\geqslant i=1}^{n-1} \frac{[h_i - r_j + j - i]!}{[r_i - r_j + j - i]!}\right)\left(\prod_{j>i=1}^{n} \frac{[h_i - h_j + j - i - 1]!}{[r_i - h_j + j - i - 1]!}\right)\right]^{1/2}. \tag{4.13}$$

This result coincides with the coefficient $\left(\tau_n(\mu_{n-1}, \mu_n)\right)^{1/2}$ of Ueno *et al* (1989).

The construction of the GT basis from its highest-weight state, given in proposition 7 of Ueno *et al* (1989), now appears as a special case of (4.8), successively applied to $u_q(n)$, $u_q(n-1), \ldots, u_q(2)$. For completeness, we state their result in the notations used in the present paper.

Theorem 4.6. Any GT basis vector $|h_{ij})$ belonging to the carrier space of a $u_q(n)$ irrep $[h_1 h_2 \ldots h_n]$ can be obtained from the highest-weight state $\left|{h_i \atop h_i}\right\rangle$ as

$$|h_{ij}) = \left(\prod_{j=2}^{n} N^{h_{1j} h_{2j} \ldots h_{j-1,j}}_{h_{1,j-1} h_{2,j-1} \ldots h_{j-1,j-1}}\right)^{-1} (L_2^1)^{h_{12}-h_{11}} \left(\prod_{m=1}^{2} (L_3^m)^{h_{m3}-h_{m2}}\right) \times \cdots$$
$$\times \left(\prod_{m=1}^{n-1} (L_n^m)^{h_{mn}-h_{m,n-1}}\right) \left|{h_i \atop h_i}\right\rangle \tag{4.14}$$

where the lowering operators are defined in (3.16) and the normalization coefficients in (4.13).

Remark. Note that in (4.14) the lowering operators belonging to distinct q-subalgebras of $u_q(n)$ are to be maintained in the order indicated.

5. Conclusion

In the present paper, we found an explicit solution to the recursion relations (3.25*a*), (3.28) and (3.31*a*) for the $u_q(n) \supset u_q(n-1)$ lowering operators, first proposed by Ueno *et al* (1989), and solved a similar problem for the corresponding raising operators, which were not considered by these authors. Our main results are contained in (3.16) and (3.17). Some special cases, corresponding to small n values, are given in (3.32)–(3.34).

The expressions found for the $u_q(n) \supset u_q(n-1)$ lowering and raising operators enable one to construct the GT basis vectors in explicit form whenever required, e.g. when dealing with applications of $u_q(n)$ to some physical models. Equation (3.33), for instance, was already used to determine a q-boson realization of the GT basis for an arbitrary $u_q(3)$ two-row irrep (Quesne 1991).

Appendix. Calculation of the q-commutators $[E_i^{i+p}, E_k^{k+r}]_{q^a}$ for $p \leqslant r$

The purpose of this appendix is to prove by a double induction over p and r the results for $[E_i^{i+p}, E_k^{k+r}]_{q^a}$, $p \leqslant r$, given in column 3 of table 1 for the values of a listed in column 2 of the latter. As a starting point corresponding to $p = r = 1$, we use (2.2*a*), as well as (2.6*a*) for $p = 2$.

For $p = 1$ and arbitrary values of $r > 1$, rows 3, 5 and 8 of table 1 disappear, while the results listed in rows 2 and 9 result from definition (2.6*a*) and from (2.7) and (2.12*a*), respectively. Next, the results in rows 1, 4, 7 and 10 are demonstrated by induction over r with the help of (2.12*a*) and (2.10) where $a = 1$, $b = c = 0$; $a = 1$, $b = 0$, $c = -1$; $a = 1$, $b = -1$, $c = 0$; and $a = 1$, $b = c = 0$, respectively. The corresponding starting values of r are $r = 1$, $r = 2$, $r = 2$, and $r = 1$.

Finally, consider the result in row 6. Its proof is more involved and is based on the fact that the equation

$$(1+q^{-1})[E_i^{i+1}, E_k^{k+r}] = 0 \qquad k < i < i+1 < k+r \tag{A1}$$

is equivalent to

$$[E_i^{i+1}, E_k^{k+r}]_{q^{-2}} = -q^{-1}[E_i^{i+1}, E_k^{k+r}]_{q^2} \qquad k < i < i+1 < k+r \tag{A2}$$

as it is easily shown by expanding all the q-commutators. Equation (A2) is now demonstrated as follows:

$$\begin{aligned}[E_i^{i+1}, E_k^{k+r}]_{q^{-2}} &= \left[E_i^{i+1}, [E_k^i, E_i^{k+r}]_q\right]_{q^{-2}} = -q^{-1/2}[E_k^{i+1}, E_i^{k+r}] \\ &= -q^{-1/2}[E_k^{i+1}, [E_i^{i+1}, E_{i+1}^{k+r}]_q] = -q^{-1}[E_i^{i+1}, E_k^{k+r}]_{q^2} \\ &\quad k < i < i+1 < k+r.\end{aligned} \tag{A3}$$

In the first step, we used (2.12a), in the second (2.10) with $a = b = 1$, $c = -1$, and the results in rows 4 and 9, in the third (2.6a), and in the fourth (2.10) with $a = 1$, $b = c = -1$, (2.7) and (2.12a), as well as the result in row 7.

Going now to arbitrary p and r values such that $1 < p \leqslant r$, we first note that the result in row 5 is obvious, while those in rows 2 and 9 directly follow from (2.12a) and (2.7). The demonstration of the results in rows 1, 4, 7 and 10 by induction over p and of that in row 6 are similar to the corresponding proofs for the case $1 = p < r$. Finally, for the case in row 3 (and similarly for that in row 8), we obtain

$$\begin{aligned}[E_i^{i+p}, E_k^{k+r}] &= [E_i^{i+p}, [E_k^{i+p}, E_{i+p}^{k+r}]_q] = q^{-1/2}[E_k^{i+p}, E_i^{k+r}]_{q^2} \\ &= -(q^{1/2} - q^{-1/2}) E_i^{k+r} E_k^{i+p} \qquad i < k < i+p < k+r \qquad \text{(A4)}\end{aligned}$$

where in the first step we used (2.12a), in the second (2.10) with $a = 1$, $b = c = -1$, (2.7), (2.12a), and the result in row 7, and in the third the result in row 6. This completes the proof of the q-commutation relations listed in column 3 of table 1.

References

Chaichian M, Ellinas D and Popowicz Z 1990 *Phys. Lett.* **248B** 95

Chakrabarti A 1991 *J. Math. Phys.* **32** 1227

Drinfeld V G 1986 *Proc. Int. Congr. of Mathematicians (Berkeley)* ed A M Gleason (Providence, RI: American Mathematical Society) p 798

Gel'fand I M and Tseitlin M L 1950 *Dokl. Akad. Nauk SSSR* **71** 825

Jimbo M 1985a *Lett. Math. Phys.* **10** 63

—— 1985b *Field Theory, Quantum Gravity and Strings (Lecture Notes in Physics 246)* (Berlin: Springer) p 335

Lusztig G 1988 *Adv. Math.* **70** 237

Nagel J G and Moshinsky M 1965a *J. Math. Phys.* **6** 682

—— 1965b *Rev. Mex. Fis.* **14** 29

Quesne C 1991 Complementarity of $su_q(3)$ and $u_q(2)$ and q-boson realization of the $su_q(3)$ irreducible representations *Preprint* Université Libre de Bruxelles PNT/15/91

Rosso M 1988 *Commun. Math. Phys.* **117** 581

—— 1989 *Commun. Math. Phys.* **124** 307

Ueno K, Takebayashi T and Shibukawa Y 1989 *Lett. Math. Phys.* **18** 215